CS 331, Fall 2025
Lecture 3 (9/3)

Today:
- Matrices
- Fibonacci
- FFT

Warmup: Stock market

prices on n days

Input: L is a list of n elements in $\mathbb{R}$

Output: (i, j) with $1 \le i \le j \le n$

must buy then sell

maximizing $L[j] - L[i]$

sell buy

Example

Day

1	2	3	4	5	6	7
8	12	4	9	17	1	13

GME stocks

How to maximize profit?

Idea 1: Try everything.

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For j ∈ [n]:
    For i ∈ [j]:        } O(n^2)
        ... # keep track of running max
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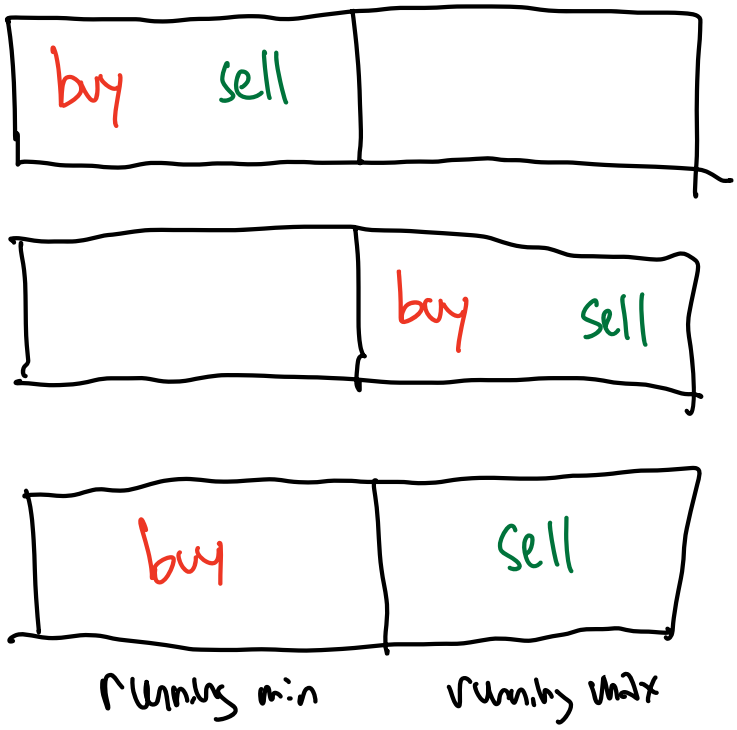
Idea 2: Divide-and-conquer

$T(n)$ = runtime on length-n input

Cases:



$T(\frac{n}{2})$

$+ T(\frac{n}{2})$

$+ O(n)$

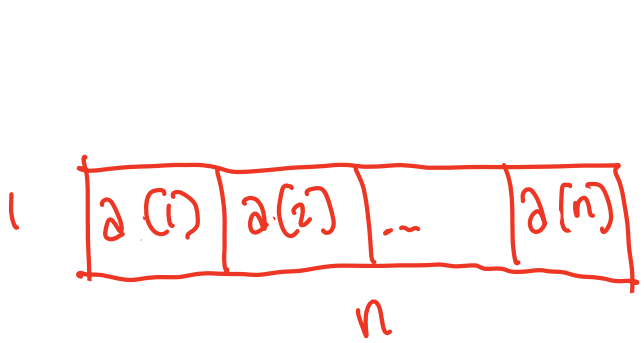
$= O(n \log(n))$

Idea 3: $O(n)$ next time ☺

Matrix multiplication (Part II, Section 6.1)

Warmup: **Vector-vector**

"inner product"
aka
"dot product"



$$n = \sum_{i \in [n]} a[i]\, b[i]$$

$O(n)$ time.

↑ linear time

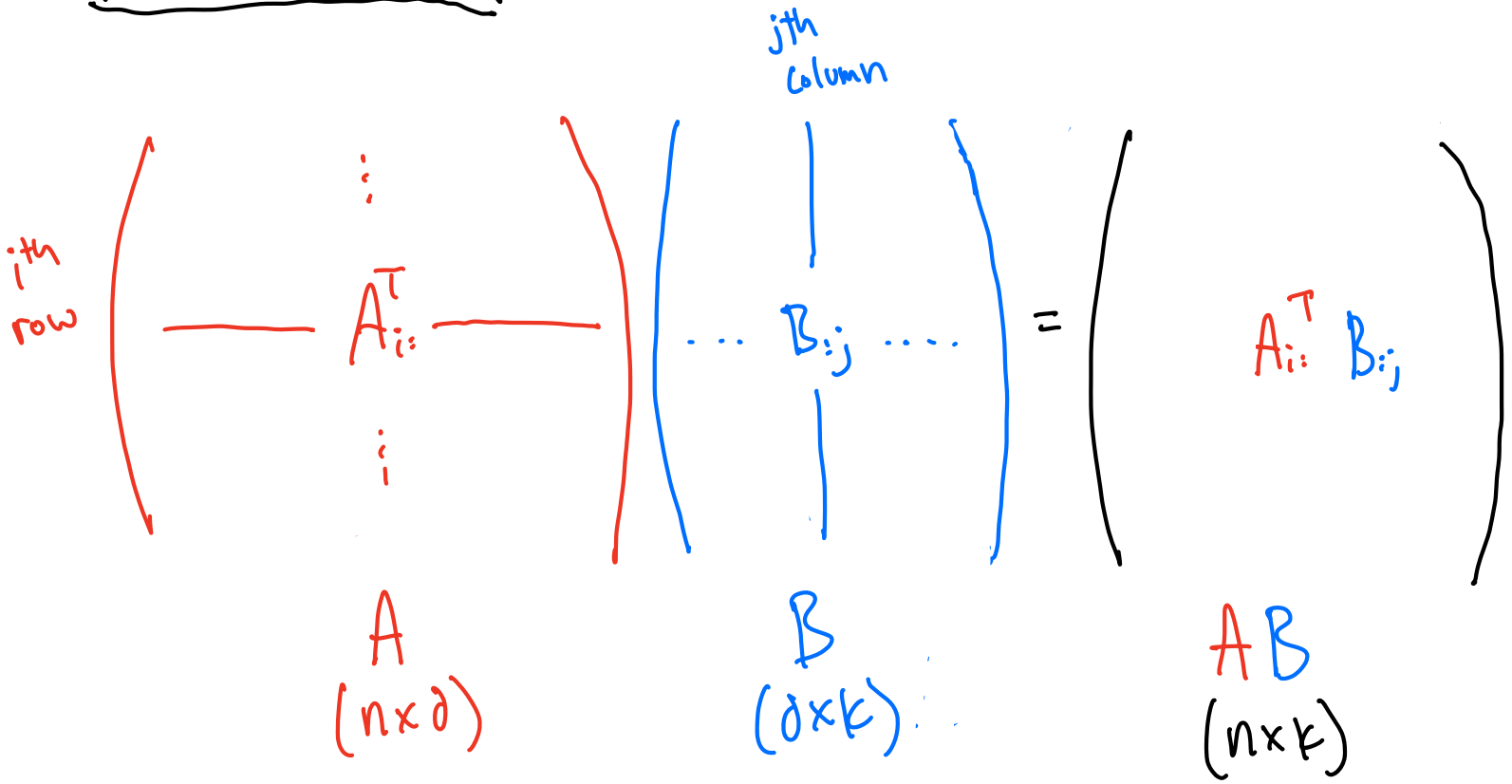
Notation:

In this class, vectors are lower-case and columns $(n \times 1)$

Dot product of $a, b \in \mathbb{R}^{n \times 1}$:

$$\langle a, b \rangle = a^T b = \sum_{i \in [n]} a(i)\, b[i]$$

Matrix-matrix: dot every row/col pair



Applications: entirety of modern ML/data science.

Intuition:

n = # examples

d = # features

Examples

$$\begin{pmatrix} 1 & 2 \\ -3 & -1 \end{pmatrix}\begin{pmatrix} 4 & 1 \\ 6 & -2 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix}\begin{pmatrix} 7 & 2 & 6 \\ 1 & 3 & 5 \\ 1 & 2 & 3 \end{pmatrix} \qquad \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$$

Naive matrix-matrix: $(n \times d)(d \times k)$

$$\underbrace{O(d)}_{\text{per dot product}} \times \underbrace{nk}_{\#\text{ entries}} = O(ndk).$$

If $n=d=k$, this is $O(n^3)$. Linear time $= O(n^2)$

Aside Block matrix multiplication works like you think!

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} = \begin{pmatrix} A_{11}B_{11}+A_{12}B_{21} & A_{11}B_{12}+A_{12}B_{22} \\ A_{21}B_{11}+A_{22}B_{21} & A_{11}B_{12}+A_{22}B_{2} \end{pmatrix}$$

$A_{11}, A_{12}, A_{21}, A_{22}, B_{11}, B_{12}, B_{21}, B_{22}$ are matrices

Intuition: Splitting dot product

$$A_{1:}^T B_{:1} = [A_{11}]_{1:}^T [B_{11}]_{:1} + [A_{12}]_{1:}^T [B_{12}]_{:1}$$

Naive recursion: $T(n) = 8T\left(\frac{n}{2}\right) + O(n^2)$

$$= O(n^3)$$

Strassen recursion: $T(n) = 7T\left(\frac{n}{2}\right) + O(n^2) = O(n^{2.807\ldots})$

World record: $T(n) = O(n^{2.3714\ldots})$ (galactic algorithm...)

When can we do better?

Matrix-vector

Let A is $n \times n$

b is $n \times 1$

(input size: $O(n^2 + n) = O(n^2)$)

We can compute Ab in time:

$$(O(n) \text{ per entry}) \times n = O(n^2)$$

Linear time ☺

Can we do better if A fixed, b is input?

→ HW1, P3

→ FFT (today)

Fibonacci (Part II, Section 6.3)

Classic interview question / "gotcha"

Input: $n \in \mathbb{N}$

Output: F_n, n^{th} Fibonacci number

Recursive definition:

$$F_0 = 1, \; F_1 = 2, \; F_2 = 3, \; F_3 = 5, \; F_4 = 8$$

$$\ldots \; F_n = F_{n-1} + F_{n-2}$$

Observation: $F_n = \exp(\Theta(n))$

Aside: In fact, $F_n \approx \varphi^n$, where $\varphi \approx 1.618$ is "golden ratio"

Naive (??): $\exp(\Theta(n))$ time

Naive: n additions... but #s are $O(n)$ digits

$\rightarrow$ $O(n^2)$ time

Improvement using matrix multiplication

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \qquad A \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y \\ x \end{pmatrix}$$

Main claim: $A^n \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} F_n \\ F_{n-1} \end{pmatrix}$

Proof (induction):

$$A^{n+1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = A \cdot A^n \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} F_n + F_{n-1} \\ F_n \end{pmatrix}$$

($F_n + F_{n-1} = F_{n+1}$)

Algo: 1) Compute $A, A^2, A^4, \ldots A^{2^{\lfloor \log_2(n) \rfloor}}$ $O(n \log(n))$

2) Write n in binary $O(\log(n))$

e.g. $57 = 32 + 16 + 8 + 1$

3) Multiply those matrix powers $O(n \log(n))$

e.g. $A^{57} = A^{32} \cdot A^{16} \cdot A^8 \cdot A$

Fast Fourier Transform (Part II, Section 6.2)

Let $n = 2^k$, $k = 1, 2, \ldots$

F_n is $n \times n$ Complex matrix.

$DFT_n(b)$: return $F_n b$ in time $O(n \log(n))$

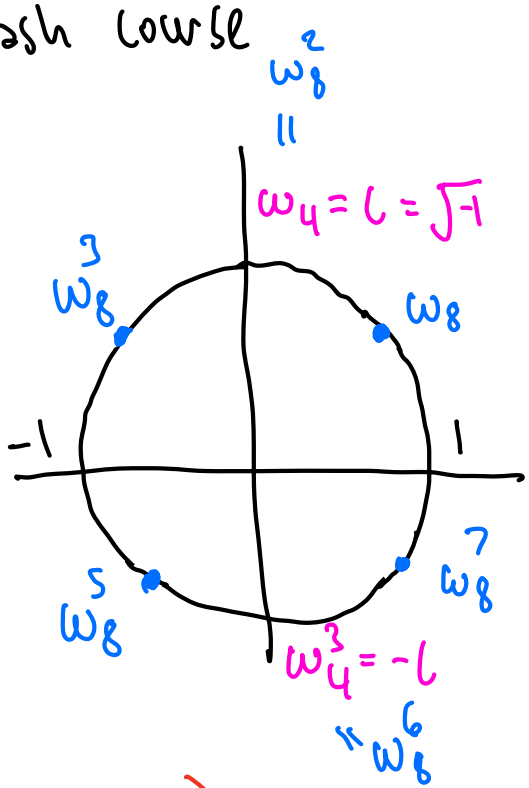
Applications: faster multiplication
signal processing (learn "spectrum" of waves)

Aside Complex numbers crash course

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

$$\omega_n = e^{i \cdot \frac{2\pi}{n}}$$ "n^{th} root of unity"

$\frac{2\pi}{n}$ = rotate $\frac{1}{n}^{th}$ of a circle



Remember: $\omega_n^n = 1$. (full lap around circle)

So, what is F_n?

$$F_n[i][j] = \omega_n^{(i-1)(j-1)}$$

Interpretation:
i^{th} row of F_n goes around unit circle, $\frac{(i-1)}{n} \cdot 2\pi$ at a time.

$$F_1 = \begin{pmatrix} 1 \end{pmatrix} \qquad \omega_1 = 1$$

$$F_2 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \qquad \omega_2 = -1$$

$$F_4 = \begin{pmatrix} \omega^0 & \omega^0 & \omega^0 & \omega^0 \\ \omega^0 & \omega^1 & \omega^2 & \omega^3 \\ \omega^0 & \omega^2 & \omega^0 & \omega^2 \\ \omega^0 & \omega^3 & \omega^2 & \omega^1 \end{pmatrix} \qquad \omega \equiv \omega_4 = i$$

$$F_8 = \begin{pmatrix} \omega^0 & \omega^0 & \omega^0 & \omega^0 & \omega^0 & \omega^0 & \omega^0 & \omega^0 \\ \omega^0 & \omega^1 & \omega^2 & \omega^3 & \omega^4 & \omega^5 & \omega^6 & \omega^7 \\ \omega^0 & \omega^2 & \omega^4 & \omega^6 & \omega^0 & \omega^2 & \omega^4 & \omega^6 \\ \omega^0 & \omega^3 & \omega^6 & \omega^1 & \omega^4 & \omega^7 & \omega^2 & \omega^5 \\ \omega^0 & \omega^4 & \omega^0 & \omega^4 & \omega^0 & \omega^4 & \omega^0 & \omega^4 \\ \omega^0 & \omega^5 & \omega^2 & \omega^7 & \omega^4 & \omega^1 & \omega^6 & \omega^3 \\ \omega^0 & \omega^6 & \omega^4 & \omega^2 & \omega^0 & \omega^6 & \omega^4 & \omega^2 \\ \omega^0 & \omega^7 & \omega^6 & \omega^5 & \omega^4 & \omega^3 & \omega^2 & \omega^1 \end{pmatrix}$$

$$\omega \equiv \omega_8 = \exp\left(i \cdot \frac{2\pi}{8}\right)$$

Recall
$\omega_8^2 = \omega_4$

Claim: $T(n) = 2T\left(\frac{n}{2}\right) + O(n)$

$T(n)$: time to compute $F_n b$

Aside

Two ways of thinking about Ab

Way 1: $(Ab)(i) = A_{i:}^T b$

Way 2:

$$\begin{pmatrix} | & | & & | \\ A_{:1} & A_{:2} & \cdots & A_{:n} \\ | & | & & | \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

$$= b_1 A_{:1} + b_2 A_{:2} + \cdots b_n A_{:n}$$

note: order doesn't matter!

Observations

T1 T2

1) $F_n b = (F_n)_{odd}\, b_{odd} + (F_n)_{even}\, b_{even}$

$n \times \frac{n}{2}$ $\frac{n}{2} \times 1$ $n \times \frac{n}{2}$ $\frac{n}{2} \times 1$

2) $(F_n)_{odd} = \begin{pmatrix} F_{\frac{n}{2}} \\ \\ F_{\frac{n}{2}} \end{pmatrix}$

Can compute T1 in: $T\left(\frac{n}{2}\right) + O(n)$ time.

3) $(F_n)_{even} = \begin{pmatrix} 1 & & & \\ & \omega_n & & \\ & & \omega_n^2 & \\ & & & \ddots \\ & & & & \omega_n^{n-1} \end{pmatrix} \begin{pmatrix} F_{\frac{n}{2}} \\ \\ F_{\frac{n}{2}} \end{pmatrix}$

"twiddle factors"

Can compute (T2) in $T(\frac{n}{2}) + O(n)$ time.

Putting it together:

$$F_n b = \begin{pmatrix} F_{\frac{n}{2}} b_{odd} \\ \\ F_{\frac{n}{2}} b_{odd} \end{pmatrix} \quad T(\tfrac{n}{2}) + O(n)$$

$$+ \begin{pmatrix} 1 & & & \\ & \omega_n & & \\ & & \omega_n^2 & \\ & & & \ddots \\ & & & & \omega_n^{n-1} \end{pmatrix} \begin{pmatrix} F_{\frac{n}{2}} b_{even} \\ \\ F_{\frac{n}{2}} b_{even} \end{pmatrix} \quad T(\tfrac{n}{2}) + O(n)$$

FFT in $O(n \log(n))$ time!

About multiplication... $O(n \log(n))$ via FFT!

$$a = a_{n-1}10^{n-1} + a_{n-2}10^{n-2} + \dots a_1 10^1 + a_0$$

$$= p_a(10) \qquad (\text{coeffs} = \text{digits of } a)$$

$$b = b_{n-1}10^{n-1} + b_{n-2}10^{n-2} + \dots b_1 10^1 + b_0$$

$$= p_b(10) \qquad (\text{coeffs} = \text{digits of } b)$$

To compute ab, just need coeffs of $p_a p_b$.

Amazing fact: $F_n \underbrace{v}_{\text{coeffs}} = \underbrace{\begin{pmatrix} p_v(1) \\ p_v(\omega_n) \\ p_v(\omega_n^2) \\ \vdots \\ p_v(\omega_n^{n-1}) \end{pmatrix}}_{\text{evaluations}}$

FFT-based multiplication:

1) Evaluate $F_n a$, $F_n b$

2) Let $c = \{p_a(x)\, p_b(x)\}$ for $x = 1, \omega_n, \omega_n^2, \dots, \omega_n^{n-1}$

3) Evaluate $F_n^{-1} c$. Gives coeffs of $p_a p_b$!

4) Evaluate $(p_a p_b)(10) = ab$